Brown University
Math 0350 Midterm
19 October 2017
Instructor: Samuel S. Watson

Name:

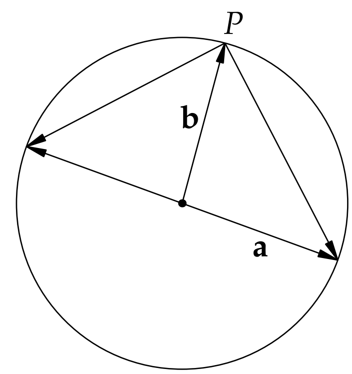
Problem 1

In this problem, we will use vectors to show that an angle formed by connecting a point on a circle to two diametrically opposite points on the same circle is a right angle.

(a) Use vector addition/subtraction/scaling to label all three unlabeled vectors in the figure below in terms of $\mathbf{a}$ and/or $\mathbf{b}$. Write your answers directly on the figure.

(b) Use part (a) to show that the dot product of the two vectors in the figure whose tails are at P is equal to zero.

Solution



Problem 2

The position of a particle at time t is given by $\mathbf{r}(t) = \langle 2t, \frac{2}{t}, -t^2 \rangle$. Find the positive time t when the velocity of the particle is perpendicular to its acceleration. You may express your answer as a radical.

Solution

Final answer:

Problem 3

Consider the six points $(-1, 1, 0)$, $(1, -1, 0)$, $(-1, 0, 1)$, $(1, 0, -1)$, $(0, 1, -1)$, and $(0, -1, 1)$.

(a) Show that these six points are coplanar. (Hint: find the plane P passing through three of them, and then show that the other three also lie in this plane).

(b) Find the distance from the plane P to the line **segment** from $(4, 3, 1)$ to $(-2, -3, 4)$. (Hint: think carefully about this one.)

Solution

Final answer:

Problem 4

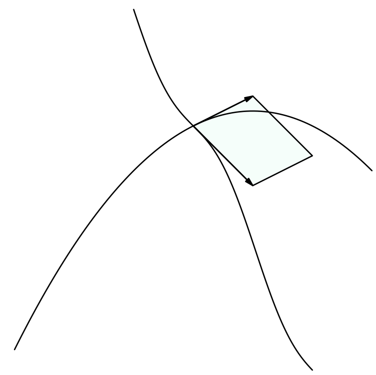
The figure below shows the two curves $\mathbf{r}_1(t) = \langle t, \frac{1}{2}t - \frac{1}{4}t^2 \rangle$ and $\mathbf{r}_2(t) = \langle t, -2t + \frac{1}{3}\sin 3t \rangle$, which intersect at the origin.

(a) Which path is $\mathbf{r}_1$ and which is $\mathbf{r}_2$? Just label them in the figure.

(b) Find $\mathbf{r}_1'(0)$ and $\mathbf{r}_2'(0)$ (these are the two vectors shown).

(c) Use the cross product to find the area of the parallelogram which has $\mathbf{r}_1'(0)$ and $\mathbf{r}_2'(0)$ as two of its sides.

Solution



Problem 5

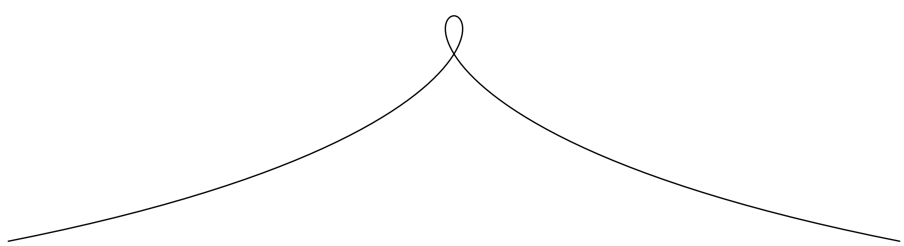
(a) Place a dot at the point of maximum curvature for the smooth path shown (no calculation!).

(b) The parametric equation for this path is $\mathbf{r}(t) = \langle 2t^3 - t, 5\cos(t) \rangle$. The path's curvature turns out to be

$$\kappa(t) = \frac{1}{\left(\left(6t^2-1\right)^2 + 25\sin^2 t\right)^2} \sqrt{25\left(6t^2-1\right)^2\left(6t^2\cos t - 12t\sin t - \cos t\right)^2 + \left(12t\left(\left(6t^2-1\right)^2 + 25\sin^2 t\right) + \left(-6t^2+1\right)\left(72t^3 - 12t + \frac{25}{2}\sin(2t)\right)\right)^2}.$$

Explain what steps you would take to arrive at the above formula for $\kappa(t)$. Just state the necessary formulas and name any intermediate quantities you would calculate; there is no need to do **any** computation.

Solution



Problem 6

Each of the 40 tentacles in this piece of Chihuly-inspired graphic art is a random rotation of the path $\mathbf{r}(t) = \langle \sin t \cos t, \sin^2 t, t \rangle$ as t varies from 0 to 2. Find the **total** length of all the tentacles.

Hint: the integrand gets pretty tidy pretty fast. If the integration step is not trivial, go back and check your work.

(Credit to Oliver Knill for this problem concept)

Solution

Final answer:

Problem 7

Region A consists of all the points in 3D space satisfying the spherical coordinate inequalities $\rho \leq 1$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq \frac{\pi}{4}$. Region B consists of all the points in 3D space satisfying the cylindrical coordinate inequalities $r \leq \frac{1}{\sqrt{2}}$, $0 \leq \theta \leq \pi$, and $0 \leq z \leq 1$. Without calculating any volumes, which region is larger? Sketch both regions carefully and explain how you can be sure one is larger than the other without calculating the volume of either.

Solution

Problem 8

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with the property that, for all real numbers m, we have $\lim_{t \to 0} f(t, mt) = 6$. Answer the following questions and **use complete sentences** to clearly explain your reasoning.

(a) Based on this information, what can be determined about the value of $f(0,0)$? (In other words, for which values of $b \in \mathbb{R}$ does there exist a function satisfying the above properties and also $f(0,0) = b$?)

(b) If the limit of $f(x,y)$ as $(x,y) \to (0,0)$ exists and is equal to L, then what is the value of L?

(c) Suppose that f is continuous. Find $\lim_{t \to 0} f(t^2, \sin t)$.

(d) Suppose that f is continuous. Explain why $\lim_{t \to 0} f(t^2, \cos t)$ cannot be determined using the given information.

Solution

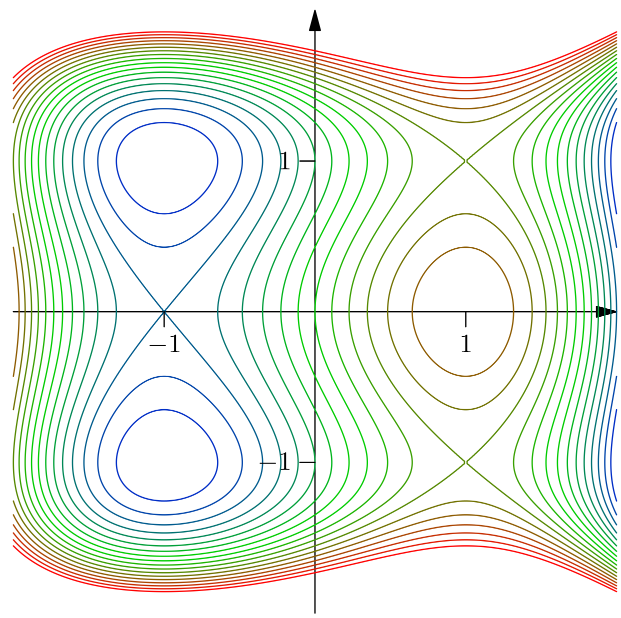
Problem 9

(a) Find the critical points of $f(x, y) = -x^3 + 3x + y^4 - 2y^2$ and place dots at those locations in the figure shown (which depicts some level curves of f).

(b) To find the maximum value of $f(x, y)$ for any point (x, y) satisfying $-2 \le x \le 2$ and $-2 \le y \le 2$, is it sufficient to find the maximum among the values of f at the critical points found in (a)? Explain why or why not.

(c) Apply the second derivative test to classify each of the critical points of f.

Solution



Problem 10

Suppose that $2x + 3y + 2z = 9$ is the equation of the plane tangent to the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(1, 1, 2)$.

(a) Rewrite the equation of the tangent plane in the form $z = a(x - 1) + b(y - 1) + c$, where a and b are numbers.

(b) Use (a) to find $(\partial_x f)(1, 1)$ and $(\partial_y f)(1, 1)$.

(c) Find $L(0.99, 1.02)$, where L is the linear approximation of f at the point $(1, 1)$.

Solution

BONUS 1 (0 points)

Find the third-order Maclaurin polynomial of

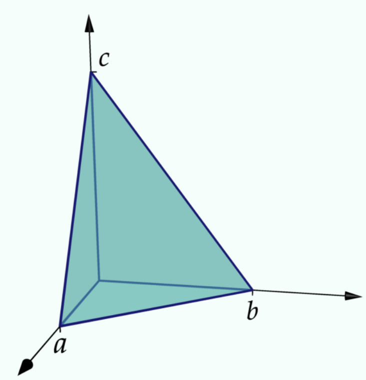
$$x^3y^3 + 6x^3y^2 - 8x^3y - 9x^3 - 4x^2y^3 - 10x^2y^2 + x^2y + 6x^2 + 8xy^3 - 4xy^2 - 2xy + 8x + 4y^3 - 9y^2 - 11$$

Solution

BONUS 2 (0 points)

Prove the 3D Pythagorean theorem, for tetrahedrons with a trirectangular vertex (that is, a vertex where all three incident faces have a right angle): *the sum of the squares of the areas of the three smallest faces is equal to the square of the area of the largest face.*

You may work a tetrahedron with vertices at the origin, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, as shown.



Solution

Additional space

Additional space