Brown University
Math 0350 Concepts List
Instructor: Samuel S. Watson

Chapter 1: Transformations of Euclidean space

(a) $\mathbb{R}^1$ is the number line, $\mathbb{R}^2$ is the plane, and $\mathbb{R}^3$ is 3D space

(b) The x-coordinate in $\mathbb{R}^2$ represents *signed distance to the y-axis*, and similarly for y

(c) The x-coordinate in $\mathbb{R}^3$ represents *signed distance to the yz-plane*, and similarly for y and z

(d) We can visualize a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ using its graph, which is a subset of $\mathbb{R}^{m+n}$

(e) The linear functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ are the ones of the form $L(x,y) = (ax+by, cx+dy)$, where a,b,c,d are real numbers

(f) It's handy to describe L by putting its coefficients in a matrix like $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(g) Linear functions can rotate, reflect, shear, project, and scale

(h) To figure out the factor by which a linear function transforms areas, we calculate the absolute value of the *determinant* of the matrix, which for a 2×2 matrix is $|ad - bc|$

(i) To find the determinant of a 3×3 or larger matrix, we expand by minors: (1) choose a row or column, (2) for each entry of that row or column, form a term by multiplying (i) that entry, (ii) the determinant of the matrix obtained by deleting the row and column of that entry times, and (iii) a $+1$ or -1 factor coming from a checkboard pattern of ± 1's starting with $+1$ in the top left, and (3) add up the resulting terms

Chapter 2: Vectors

1. **Vectors**

   (a) Vectors are ordered pairs or triples of real numbers which we visualize as an arrow which we are free to translate, and they are added together or multiplied by scalars componentwise

   (b) Vector operations satisfy the properties you'd expect: associativity, commutativity, distribution of scalar multiplication across vector addition, etc. These can all be proved by writing everything in terms of components

   (c) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ form a triangle, as do $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$

2. **Dot products**

   (a) To calculate the dot product of two vectors, you multiply corresponding components and add up the results

   (b) The length of a vector is $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

   (c) Dot products distribute across vector addition

   (d) $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

   (e) These properties can be used in conjunction to give simple proofs of geometric results involving orthogonality: for example, that the diagonals of a parallelogram are orthogonal if and only if the parallelogram is a rectangle

3. **Cross products**

   (a) The cross product of two vectors $\langle u_1, u_2, u_3 \rangle$ and $\langle v_1, v_2, v_3 \rangle$ can be obtained by expanding this expression along the first row:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(b) $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and is oriented so that you can curl the fingers of your right hand from $\mathbf{u}$ to $\mathbf{v}$ with your thumb in the direction of $\mathbf{u} \times \mathbf{v}$

(c) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

(d) The volume of the parallelepiped spanned by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is called the *triple scalar product* and is equal to the absolute value of $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ and is also equal to the absolute value of the determinant of the 3×3 matrix obtained by putting the components of the three vectors in order into the three rows

Chapter 3: Three-dimensional geometry

1. **Lines and planes in space**

(a) The line passing through (x_0, y_0, z_0) parallel to the vector (a, b, c) is described parametrically as the set of all points of the form $(x_0 + at, y_0 + bt, z_0 + ct)$, where t is any real number

(b) The plane passing through (x_0, y_0, z_0) and perpendicular to the vector (a, b, c) has equation

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0,$$

or

$$ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$

(c) To find the distance between any two straight figures (2D planes, 1D lines, or 0D points), some helpful ideas: find some points on the two figures, draw a right triangle involving those points and the desired distances, write down a trig expression for the desired distance, and rewrite the resulting expression in a way that substitutes dot or cross products for trig functions

2. **Vector-valued functions**

(a) The position of a particle moving in space is given by a vector-valued function of t denoted $\mathbf{r}(t)$

(b) The velocity of the particle is $\mathbf{v}(t) = \mathbf{r}'(t)$, where the derivative is taken componentwise

(c) The velocity vector is tangent to the path traced out by $\mathbf{r}(t)$, and its magnitude is the *speed* of the particle

(d) The acceleration of the particle is $\mathbf{a}(t) = \mathbf{r}''(t)$, where the derivatives are taken componentwise

(e) The arclength of a path $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is given by $\int_a^b |\mathbf{r}'(t)|\, dt$

(f) The unit tangent vector of a path $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is given by $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$.

(g) The curvature $\kappa(t)$ is given by $\frac{|d\mathbf{T}/dt|}{|\mathbf{r}'(t)|}$

3. **Quadric surfaces**

(a) The graph of an equation in three variables x, y, z is the set of all points (x, y, z) that satisfy the equation

(b) The main tool for sketching 3D surfaces is to take *traces*, which are intersections of the surface with planes, usually parallel to the coordinate planes

(c) To find the surface of revolution of curve in the z-x plane about the z-axis, replace x with $\sqrt{x^2 + y^2}$ in the equation describing the curve

(d) Quadric surfaces are the surfaces you get from graphing quadratic equations in x, y, z (like conic sections in 2D)

(e) Quadric surfaces include paraboloids (with no z^2 term), both elliptic and hyperbolic (saddle), as well as elliptic cones and one-sheeted and two-sheeted hyperboloids

4. **Cylindrical and spherical coordinates**

(a) Cylindrical coordinates represent a point (x, y, z) in space in terms of the quantities r, θ, and z, where r is distance to the z-axis, θ is the angle of (x, y) with respect to the positive x-axis, and z is the signed distance to the x-y plane

(b) Spherical coordinates represent a point (x,y,z) in space in terms of the quantities ρ, θ, and ϕ, where ρ is distance to the origin, θ is the angle of (x,y) with respect to the positive x-axis, and ϕ is the angle with respect to the positive z-axis

(c) You can work out the translations between Cartesian and spherical coordinates by drawing a pair of right triangles involving $(0,0,0)$, $(x,0,0)$, $(x,y,0)$, and (x,y,z) and using basic trigonometry

Chapter 4: Differentiation

1. **Limits and continuity**

   (a) For a function $f(x,y)$, we say that the limit of $f(x,y)$ as $(x,y) \to (a,b)$ equals L if for all $\epsilon > 0$, there is a $\delta > 0$ so that whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, the value of $f(x,y)$ is within ϵ of L.

   (b) We can check the limit of $f(x,y)$ as $(x,y) \to (a,b)$ along the line of angle θ passing through (a,b) by substituting $(x,y) = (a + t\cos\theta, b + t\sin\theta)$ and taking $t \to 0$.

   (c) If the limits along two different paths through (a,b) are not equal or if there's any path along which the limit fails to exist, then the limit of $f(x,y)$ as $(x,y) \to (a,b)$ does not exist

   (d) If the limits along all *lines* exist and are the same, then it's still possible the limit does not exist

   (e) To show that a limit equals L, you can substitute $a + r\cos\theta$ for x and $b + r\sin\theta$ for y and show that the resulting expression is close to L whenever r is small *regardless of the value of* θ. This is true whenever $|f(a + r\cos\theta, b + r\sin\theta) - L|$ can be expressed as the product of a function tending to 0 as $r \to 0$ and a bounded function of θ

   (f) Alternatively, if the function in question is continuous at (a,b) (for example, because it is a composition of functions known to be continuous), then the limit exists and equals the value of the function at (a,b)

2. **Partial derivatives**

   (a) The partial derivative $\partial_x f$ of a function $f(x,y)$ at a point (x_0,y_0) is the slope of the graph of f in the x-direction at the point (x_0,y_0). In other words, it's the slope of the trace of the graph in the plane $y = y_0$

   (b) To find the partial derivative with respect to x, we hold y constant and differentiate as usual, and to find the partial derivative with respect to y, we hold x constant and differentiate with respect to y

3. **Tangent planes and linear approximation**

   (a) If a function's graph has a tangent plane at a particular point, then the function is said to be differentiable at that point

   (b) A function which is differentiable everywhere in its domain is said to be differentiable

   (c) The equation of the plane tangent to the graph of a differentiable function f at the point $(a,b,f(a,b))$ is given by
   $$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

   (d) The above equation can be used to say how f behaves for values of (x,y) very close to (a,b): the output changes by the x-change $x - a$ times f's sensitivity to changes in x (namely $f_x(a,b)$) *plus* the y-change times f's sensitivity to changes in y (namely $f_y(a,b)$)

4. **Taylor's theorem**

   (a) The kth order Taylor polynomial of a k-times differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ centered at (a,b) is the kth-order polynomial P_k which matches as many derivatives of f as possible (a,b) (which is all of them up to order k)

   (b) P_k is given by the sum of all terms of the form $\frac{1}{i!j!}\partial_x^i \partial_y^j P_k (x-a)^i (y-b)^j$, where $i + j \le k$.

   (c) The difference between f and P_k goes to zero faster than $|\langle x,y\rangle - \langle a,b\rangle|^k$ as $(x,y) \to (a,b)$.

   (d) The linearization of f, namely $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, is the first-order Taylor polynomial of f

5. **Multivariate optimization**

   (a) The extreme value theorem guarantees that a continuous function on a closed and bounded region realizes an absolute maximum and an absolute minimum

   (b) *Critical points* of a function are points in the interior of the function's domain where both partial derivatives are equal to zero, or else where the function is not differentiable

   (c) To find the absolute min/max of a function over a given domain, we (i) find the min/max of the function on the boundary of the domain (which reduces to one or several single-variable optimization problems, since we can parametrize the pieces of the boundary), (ii) find all critical points inside the domain, and (iii) select the largest and smallest values.

6. **Second derivative test**

   (a) We can sometimes detect whether a critical point is a local min or max using second derivatives

   (b) We evaluate $D = f_{xx}f_{yy} - f_{xy}^2$ at the critical point

   (c) If D is positive, the function has a local min or max, and if it's negative, the function has a saddle point

   (d) To remember the classification rules, think of the basic examples $x^2 + y^2$, $x^2 - y^2$, and $-x^2 - y^2$

7. **Directional derivatives and gradient**

   (a) The directional derivative $D_{\mathbf{u}}f$ of a function f in a direction $\mathbf{u}$ at a point (a, b) is defined to be the rate of change of f as x and y move jointly away from (a, b) in the $\mathbf{u}$ direction. Here $\mathbf{u}$ is understood to have unit length

   (b) The gradient ∇f of a function f is obtained by putting all the partial derivatives of a function f together into a vector

   (c) The directional derivative in the $\mathbf{u}$ direction is equal to $\nabla f \cdot \mathbf{u}$

   (d) Because of the formula $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$, the direction of the gradient is the direction in which f increases most rapidly. The direction opposite to the gradient is the direction of maximum decrease, and the directions orthogonal to these are the ones in which f is constant

   (e) It follows that ∇f at each point is orthogonal to the level set of f through that point (since the directions tangent to the level set are directions of no change, and hence zero directional derivative)

8. **Chain rule**

   (a) If we compose a function $\mathbf{r} : \mathbb{R}^1 \to \mathbb{R}^2$ with a function $f : \mathbb{R}^2 \to \mathbb{R}^1$, we get a function whose derivative is

   $$(f \circ \mathbf{r})'(t) = (\nabla f)(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

   (b) The chain rule and directional differentiation and linearization are just different ways of saying the same thing: to find how much f changes when we make a small move from $\mathbf{r}(t)$ to $\mathbf{r}(t) + \mathbf{r}'(t)\Delta t$, we dot the gradient of f with the small step $\mathbf{r}'(t)\Delta t$

9. **Lagrange Multipliers**

   (a) Let $n \geq 2$. To find critical points of the restriction of a function $f : \mathbb{R}^n \to \mathbb{R}$ to a level set of a function g, we solve the system

   $$\nabla f = \lambda \nabla g$$

   together with the given constraint equation $g(x, y) = c$ or $g(x, y, z) = c$

   (b) The equation $\nabla f = \lambda \nabla g$ holds at any maximum because if ∇f were not orthogonal to the level set of g at a given point, we could move along the level set in a direction which has a positive dot product with ∇f. This would increase the function a bit. And same idea for minima.

   (c) If a constraint is specified using distances, it's usually easier to write the constraint equation using *squared* distance rather than distance

Chapter 5: Integration

1. **Integration in two dimensions**

   (a) The integral of a function $f(x, y)$ over a 2D region R is the signed volume of the 3D region between the graph of f and the xy-plane. *Signed* means that volumes below the xy-plane count as negative

   (b) To find the integral of a function $f(x, y)$ over a 2D region R, we set up a double iterated integral over R: the bounds for the outer integral are the smallest and largest values y can be for any point in R, and the bounds for the inner integral are the smallest and largest values x can be for any point in *each "y = constant" slice* of the region. We can also do it with x and y switched, and we get the same value for the integral

   (c) Sometimes a directly-impossible iterated integral can be solved by rewriting it as a double integral and then writing it as an iterated integral with the order of x and y switched

   (d) If the region R has some polar symmetries, the iterated integral can be set up in polar coordinates instead. This is the same except (i) the area is $r\,dr\,d\theta$ instead of $dx\,dy$, and (ii) you'll need to substitute $x = r\cos\theta$ and $y = r\sin\theta$ to express the function's values in terms of r and θ

2. **Integration in three dimensions**

   (a) The integral of a function f defined on a 3D region represents the mass of a solid occupying that region and having density f at each point in the region (here we need to interpret negative values of f as contributing "negative mass")

   (b) Iterated integrals follow the same story as in 2D: for the order $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, the bounds for the outer integral are the smallest and largest values of z for any point in the region of integration, then the bounds for the middle integral are the smallest and largest values of y for any point in the region in each "z = constant" plane, and the inner bounds are the smallest and largest values of x for any point in the region in each "(y, z) = constant" line

   (c) Keep in mind that the bounds of integration can depend only on variables which appear in integrals *farther outside*. So if your integration order is $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, then the inner integral's bounds can depend on y and/or z, the middle integral's bounds can depend only on z, and the outer integral's bounds must be constant

3. **Integration in cylindrical and spherical coordinates**

   (a) Same idea, but the volume elements are $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$ and $\mathrm{d}V = \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

   (b) Integration in these coordinate systems can be handy when the region of integration **or the integrand** can be conveniently expressed using cylindrical or spherical coordinates

4. **Multivariate change of variables (custom coordinates)**

   (a) If we need to integrate over a funky region, we write that region as the image under some transformation $(x(u, v), y(u, v))$ of some simpler region in uv-space

   (b) The original integral can be expressed as an integral over the simpler region; this requires that we (i) express the integrand in terms of u and v and (ii) multiply by the **Jacobian**, which is the absolute value of the determinant of the matrix of partial derivatives of all four partial derivatives of x or y with respect to u or v

   (c) To find the transformation, it helps to write the boundaries of the region of integration as level sets of some functions of x and y. We call these u and v and solve the resulting system of equations for x and y

   (d) We can often avoid having to solve for x and y by using the identity

   $$\left|\frac{\partial(x, y)}{\partial(u, v)}\right|^{-1} = \left|\frac{\partial(u, v)}{\partial(x, y)}\right|$$

5. **Applications**

   (a) The mass of a solid whose mass density at each point is given by a function ρ is equal to the integral of ρ over the region occupied by the solid

(b) The x-coordinate of the center of mass of a solid with mass density ρ is equal to

$$\frac{\iiint x\rho(x,y,z)\,dV}{\iiint \rho(x,y,z)\,dV},$$

where both integrals are over the region occupied by the solid. Same for y and z

(c) The average value of any function f defined on a region is equal to the integral of f over that region divided by the integral of 1 over that region

(d) The moment of inertia of a solid about an axis is equal to the integral of the solid's density function times the squared distance to the axis over the region occupied by the solid

(e) A non-negative function which integrates to 1 is called a *probability density function* (pdf)

(f) We model random experiments using pdfs, with the understanding that the integral of the pdf over a region is supposed to give the probability that the outcome of the random experiment lies in that region

(g) If g is a function on the domain D of a pdf f and P is the outcome of a random experiment with pdf f, then $g(P)$ is called a **random variable** and its **expected value** is defined to be the probability-weighted average of g: $\int_D gf\,\mathrm{d}A$

Chapter 6: Vector Calculus

1. **Line integrals**

(a) A function $\mathbf{F}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^3$ is called a *vector field*, and we visualize it by drawing at each location (interpreted as input to the function) a small vector (interpreted as the output). The directions of these arrows should be faithful, but the lengths must often be shrunk to keep the figure readable

(b) The line integral of $\mathbf{F}$ along an oriented curve C with parametrization $\mathbf{r}(t)$ where t ranges from a to b can be calculated as

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\,\mathrm{d}t$$

It represents the work it takes to move a particle in the presence of a force $\mathbf{F}$ along the path, where motion with the field counts as positive and against the field as negative. It can be thought of as a measure of **overall alignment** between the oriented curve and the vector field

(c) Line integrals are *parametrization independent*, which means that we get the same answer no matter which parametrization $\mathbf{r}$ we use for a given curve (i.e., speeding up or slowing down along different parts of the curve doesn't change the answer)

(d) Line integrals are *not*, in general, path independent. This means that for some vector fields $\mathbf{F}$, the integral may be different along two different curves connecting the same pair of points

(e) A vector field $\langle M, N\rangle$ is

(i) **conservative** if it is path independent,

(ii) a **gradient** field if it is the gradient of some function f, and

(iii) **irrotational** if $N_x = M_y$

(f) The relationship between these properties is as follows:

(i) A field is conservative if and only if it's a gradient field

(ii) Any conservative field is irrotational

(iii) Any irrotational field which is differentiable on a domain with no holes is conservative

(g) The field $\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$ is irrotational but not conservative (note the hole at the origin)

2. **Fundamental theorem of calculus for line integrals**

(a) The theorem: $\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$, where C goes from $\mathbf{a}$ to $\mathbf{b}$

(b) Implications: (1) line integrals for gradient fields are path independent, (2) line integrals of gradient fields around closed curves are zero, and (3) to evaluate a line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, we can look for a function f whose gradient equals $\mathbf{F}$ and subtract its values at the endpoints (but this function might not exist)

3. **Green's theorem**

(a) The line integral of $\mathbf{F} = \langle M, N \rangle$ around a counterclockwise, simple, closed curve in the plane is equal to $\iint_R (N_x - M_y)\, dA$, where R is the region inside the curve

(b) This implies that the area of a region in the plane is equal to $\int_C \langle 0, x \rangle \cdot \mathrm{d}\mathbf{r}$, where C is the boundary of the region

4. **Surface integrals and flow**

(a) Just as a parametrized curve in 3D is a map from $\mathbb{R}^1$ (or a subset thereof) to $\mathbb{R}^3$, a parametrized surface is a map from $\mathbb{R}^2$ (or a subset thereof) to $\mathbb{R}^3$

(b) Parametrizing a surface amounts to finding a way to describe a location on the surface using a pair of numbers. Usually (in this class) a surface can be parametrized using two of the seven coordinate functions x, y, z, r, θ, ρ, ϕ

(c) An *oriented* surface is a surface together with a choice of direction from one side to the other

(d) The *flow* of a vector field $\mathbf{F}$ through an oriented surface S represents the rate of fluid movement through the surface in the given direction, if we interpret $\mathbf{F}$ as describing velocities of some fluid particles. Negative flow means that the net flow is in the direction opposite to the surface's orientation

(e) The flow is written as $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A}$, which is shorthand for $\iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}A$, which in turn is shorthand for: break the surface S into a zillion little patches, and for each patch compute the value of $\mathbf{F}$ there dotted with the unit vector $\mathbf{n}$ which is normal to the surface there (and which points in the direction specified by the surface's orientation), and multiply that by the area $\mathrm{d}A$ of the patch.

(f) The flow can be calculated using a parameterization $\mathbf{r} : D \to \mathbb{R}^3$ as

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA,$$

where $\mathbf{r}_u \times \mathbf{r}_v$ combines the Jacobian $|\mathbf{r}_u \times \mathbf{r}_v|$ and the normal vector $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

(g) A scalar function defined on a surface S can likewise be integrated using the formula

$$\iint_S f\, \mathrm{d}A = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v|\, \mathrm{d}A,$$

where $\mathbf{r} : D \to \mathbb{R}^3$ is a parametrization of S

5. **Divergence and Curl**

(a) The divergence of a vector field $\mathbf{F} = \langle M, N, P \rangle$ is defined to be $M_x + N_y + P_z$

(b) The physical interpretation of divergence at a point is net flow per unit volume out of a small ball around the point (where we're thinking of the vector field as describing the velocity of a fluid flow)

(c) The curl of a vector field $\mathbf{F} = \langle M, N, P \rangle$ is defined by expanding the "determinant"

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix},$$

where ∂_x is an abbreviation for $\frac{\partial}{\partial x}$

(d) The physical interpretation of the x-component of curl at particular point, again thinking of the vector field as representing fluid flow, is the **circulation density** around $\mathbf{i} = \langle 1, 0, 0 \rangle$. Here *circulation* means *line integral around a closed loop*. In other words, if C_r is a circular loop which surrounds a point P and is orthogonal to $\mathbf{i}$ then

$$((\nabla \times \mathbf{F})(P)) \cdot \mathbf{i} = \lim_{r \to 0} \frac{\int_{C_r} \mathbf{F} \cdot \mathrm{d}\mathbf{r}}{\text{area}(C_r)}$$

(the orientation of C_r is counterclockwise as viewed from the head of $\mathbf{i}$). And similarly for y and z

(e) $(\nabla \times \mathbf{F}) \cdot \mathbf{i}$ can be thought of as the angular velocity that $\mathbf{F}$ would induce on a small paddlewheel whose axis of rotation is parallel to $\mathbf{i} = \langle 1, 0, 0 \rangle$, and similarly for $\mathbf{j}$ and $\mathbf{k}$

(f) The curl bears the same relation to circulation density around an arbitrary axis as the gradient bears to the directional derivative in an arbitrary direction: for any unit vector $\mathbf{u}$, we have

$$(\nabla \times \mathbf{F}) \cdot \mathbf{u} = \lim_{r \to 0} \frac{\int_{C_r} \mathbf{F} \cdot \mathrm{d}\mathbf{r}}{\operatorname{area}(C_r)},$$

where C_r is a loop of radius r which is orthogonal to $\mathbf{u}$ and is oriented counterclockwise as viewed from the head of $\mathbf{u}$. (The curl is understood to be evaluated at center of C_r)

6. **Divergence theorem**

(a) The divergence theorem says if S is a closed surface (meaning it has no boundary, like a sphere or donut, unlike a hemisphere or half-plane), the flow (from the inside to outside) of any vector field $\mathbf{F}$ through S can be calculated using

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \iiint_E \nabla \cdot \mathbf{F} \, \mathrm{d}V,$$

where E is the 3D region enclosed by the surface.

(b) Since $\nabla \cdot \mathbf{F}$ represents *net flow density*, the divergence theorem is the unsurprising fact that integrating net flow density gives the total net flow

7. **Stokes' theorem**

(a) Stokes' theorem says that if the vector field being integrated is the curl of another vector field, then the flow equals:

$$\iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{A} = \int_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where ∂S is the boundary of the surface (the set of all points where if you zoom in around the point, the surface in the vicinity of that point looks like a half-plane rather than like a whole plane; for the upper hemisphere, the boundary is the equator)

(b) The orientation of ∂S is the direction for which the surface is on your left as you walk around the boundary in that direction. So, for the upper hemisphere oriented from inside to outside, the appropriate orientation of the equal would be east

(c) The left-hand side of Stokes' theorem represents *integrated circulation density*, so Stokes' theorem is just saying that integrating circulation density gives the total circulation

(d) Stokes' theorem implies that if a surface S is closed, then the flow through the surface of the curl of any vector field is equal to 0