MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Interphase Calculus III Exam I

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1. (a) (8 points) Suppose that p is a real number. Find the fourth-order Taylor polynomial of $f(x) = (1+x)^p$ centered at $x = 0$. Express your answer in terms of p.

(b) (6 points) Use your answer to part (a) to find the fourth-order Taylor polynomial of $g(x) = \dfrac{1}{\sqrt{1-x^2}}$ centered at $x = 0$. (Hint: first find the Taylor series for h where $h(y) = 1/\sqrt{1-y}$ and then substitute $y = x^2$.)

2. (10 points) Find $\int \frac{1}{x^2(1+x)}\,dx$

(b) (5 **bonus** points) Find

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{19\cdot 20}$$

(Note: your answer will be simpler than $\sum_{n=1}^{20}\frac{1}{n^2} = 17299975731542641/10838475198270720$!)

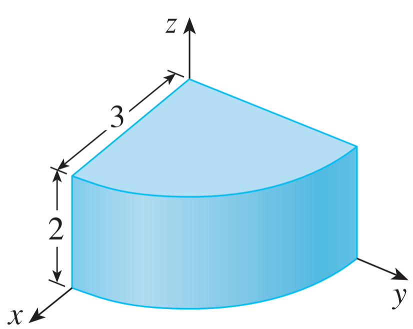
Using partial fractions, we get

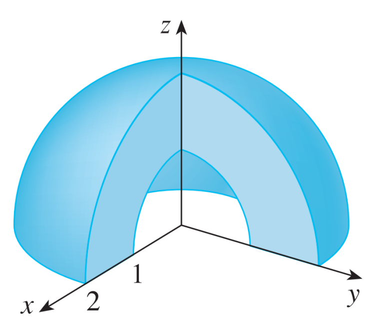
$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{19\cdot 20} &= \frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{19}-\frac{1}{20}\\ &= 1-\frac{1}{20}\\ &= \frac{19}{20}.\end{aligned}$$

3. (a) (5 points) Let S_1 be the surface defined by the spherical-coordinate equation $\theta = \pi/4$. Let S_2 be the solid defined by the spherical-coordinate inequality $1 \leq \rho \leq 4$. Sketch the intersection of S_1 and S_2.

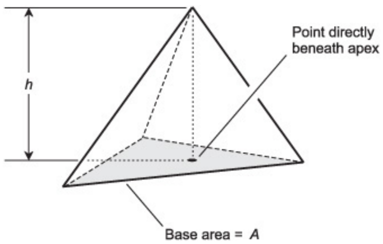
(b) (5 points) Write the **spherical-coordinate** equation of the infinite cylinder with unit radius whose axis is the z-axis.

(c) (5 points) Write systems of inequalities describing each of the following solids. In each case, use whatever coordinate system is most suitable.





4. The volume of a tetrahedron is equal to $1/3$ times the area A of its base times its height h (see the figure below).



The four planes $x + y = 1$, $x + z = 1$, $y + z = 1$, and $x + y + z = 1$ divide $\mathbb{R}^3$ up into several regions, one of which is a tetrahedron T.

(a) (5 points) Find the vertices of T.

(b) (7 points) Find the area of any one the faces of T. (Note: if you are unable to solve part (a), you may solve (b) and (c) in terms of the answer to (a).)

(c) (8 points) Find the distance from the area you found in part (b) to the vertex which is not contained in that face. Use your answer and the answer to (b) to find the volume of T.

5. The matrix

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

represents a three-dimensional rotation.

(a) (7 points) Based on the fact that M represents a rotation, what is $\det M$? Explain in words.

(b) (6 points) Confirm your answer to part (a) by directly calculating $\det M$.

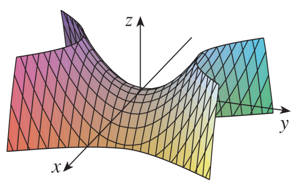
6. The location of particle A at time $t \in \mathbb{R}$ is given in Cartesian coordinates by (t, t^2, t^3). The location of particle B at time t is given in Cartesian coordinates by $(1+2t, 1+6t, 1+14t)$.

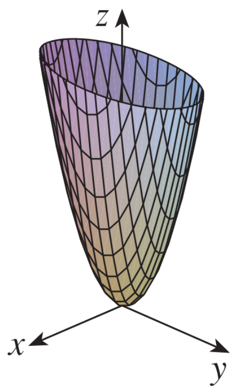
(a) (6 points) Do particles A and B collide? That is, is there a time $t \in \mathbb{R}$ at which the two particles occupy the same point in space?

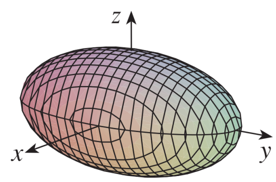
(b) (6 points) Do the paths of particles A and B intersect? That is, are there any points in space which are visited by both particles?

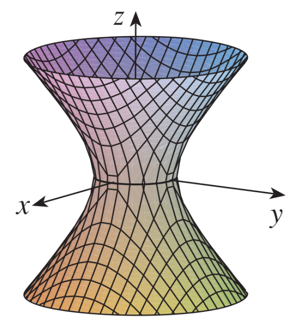
7. (a) (8 points) Matching: draw lines connecting each equation to its graph.

$x^2 + \frac{y^2}{9} + z^2 = 1$ $\qquad$ $x^2 + y^2 - z^2 = 1$ $\qquad$ $z = -x^2 + y^2$ $\qquad$ $z = x^2 + \frac{y^2}{4}$









(b) (8 points) Show that if the point (a, b, c) lies on the hyperbolic paraboloid $z = y^2 - x^2$, then the line with parametric equation $(a + t, b + t, c + 2(b - a)t)$ lies on the hyperbolic paraboloid. (Thus, even though the paraboloid is curvy, it contains lots of lines!)